other division? For example, after a year's study of algebra, plane geometry is taken up. Many of the proofs in plane geometry may be expressed very neatly in algebraic form; also many propositions in geometry open up a field of problems for algebraic solution. Now it is well known that with many students it is most difficult, if not impossible, to make them use their algebra in this work. They seem to regard the algebra as if it were finished and laid aside, and not as a vital part of their mental training, to be used everywhere that it will be of advantage.

Recently, however, even in elementary algebra considerable use has been made of the graph in explaining results and in forecasting what may be expected as the answer to any problem. It may be noted in passing that, if the student is to derive the most benefit from this work, care should be taken to prevent him from regarding the graph as merely an interesting appendix at the end of the problem, but to make him think of it as an integral part of the solution. Here analytic geometry has aided algebra in the solution of its problems in a very valuable manner, and at the same time the barriers between these two branches of mathematics have been partially broken down, although the student, of course, does not appreciate this fact.

Again, when any single division of mathematics is developed by itself, such development is apt to be narrow and even to give a wrong impression. For example, analytic geometry may be studied without any use of the calculus, and a very compact and interesting body of methods and results may be put together. But the difficulty of dealing with any but the simplest curves before the methods of calculus are used has made analytic geometry and conic sections almost synonomous terms in the past. Moreover, the student's knowledge of the meaning of analytic geometry will in this way be defective, unless he goes on to the study of calculus, and he will be obliged throughout his life to use poorer tools than is necessary. For example, tangents and normals to curves are found by methods which are intrinsically those of calculus, but which can be applied only in the simplest cases, and even then with considerable superfluous labor, because of the lack of the notation of calculus. If it should be argued that the student would have two methods, *i. e.*, the method used in analytic geometry, and the method of calculus, the reply is the question: "Does any one, after he knows how to determine tangents and normals by calculus, ever resort to the longer and more laborious methods he used in analytic geometry?"

While the study of each part of mathematics by itself will give the student a good idea of the development of thought in that particular field—a point which may be regarded by some as of decided advantage—such classification of mathematics may be perfectly well postponed to a later date, when the student has a certain body of mathematics which he knows and wishes to classify. At this later date, if he is interested, the student may with considerable pleasure and profit study the historical development of each part of his subject, for he will then have the necessary mathematical

maturity, and having, moreover, knowledge of more than one field he can compare the advances in the various fields, and note how an advance in one field made possible an advance in another field.

Mathematics is a science, an old and fundamental science, which is being used more and more in the advancement of other sciences. But this increased use in other sciences can give us little aid in determining a method of teaching mathematics—in fact, a teacher keeping this view alone in mind might meet the same kind of failure that the mere utilitarian meets, and fail quite as completely to achieve his object. It would seem, then, as if we must study mathematics as a science in itself, but find some method of obviating difficulties such as have been noted above. In looking for a new method we will first note the attitude of the professional mathematician toward his own science. As he proceeds in the study of his subject he usually finds that some type of the work appeals to him especially and devotes the greater part, if not all, of his energy to that field. But this specialization would not be apt to take place till after the first course in calculus. Before that time, he may, to be sure, be more interested in the development of the theory than in its application to problems, or vice-versa. The main thing to be noticed, however, is that *he always tries to make his solution of any problem, theoretical or practical, as direct and simple as possible, and pays little or no attention to the particular field, or fields, from which he takes his method.* If, then, the mathematician does not permit himself to be restricted to any one field for his method, why should we teach mathematics in the old divisions, the boundary lines of which have proved difficult for so many students to cross? Why set up any boundary at all for the student? Teach him his mathemetics as a unit, the only way in which he will use it, and promise him at a later date to classify it as he wishes.

In co-operation with others the writer has recently had the opportunity of laying out and conducting such a unified course of mathematics for the first two and one-half years of an engineering school. The course takes the place of distinct courses which were formerly given in theory of eqations, plane and solid analytic geometry, differential calculus, integral calculus, and differential equations. The results with the classes have fully justified the experiment, and with each succeeding year the writer's belief in the desirability and the effectiveness of such courses has increased.

In arranging a unified course the order is *an order of difficulty in the problems*, and the theory is developed as rapidly as it is required for the solution of the problems. The power of analysis and the ability to do formal work can be as logically developed in one order as in another. In addition there is the decided advantage that the same method may occur several times with problems of increasing difficulty, and by this repetition a method, at first vague, may become clear and powerful. For example, the tangent to a curve is first found for a curve of the type

$$y=a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n,$$

where the right hand member is a rational algebraic polynomial of degree n, simple numerical cases being taken first; then for a curve of which the equation is algebraic, but more complex; and finally for curves of which the equations are transcendental. The first case requires only the formula for differentiating the algebraic polynomial; before the second case can be studied, the differentiation of any algebraic function must be studied; and before the last case can be studied, the differentiation of the transcendental functions must be considered. Here the whole subject of differentiation has been developed at the same time that a use for it has been shown, and the problem of the tangent has been studied at three different times with problems arranged in an ascending order of difficulty. Of course other problems would be carried on in this same manner with the development of the formulas of differentiation.

In a unified course there may be a saving of time, as no problem will be taken up till the best method of work has been developed for that problem. But such economy of time ought not to be regarded as the chief end to be attained by such a course. If it is so regarded, the course may be open to the criticism that some parts of mathematics, well and thoroughly studied before, may be slighted in the new course. As a matter of fact everything in the old courses, if desired, may be retained in the new course, and receive quite as much attention as in the old order, though one cannot but feel that some things as ordinarily given in the old courses could be omitted with no serious loss.

Again, the unified course, if too rapid advance is attempted, may be thought to make too great demands upon a student's power of assimilating new material. But it is in this very respect that the unified course is notably superior to the older style of course. In the latter course the particular new point is dwelt upon at such length in the effort to make the student understand it thoroughly before going on, as he will not take it up again, that he may find his work monotonous, and lose his interest. In the unified course, however, he takes up the new idea, studies it in an elementary way, and learns as much of it as possible without having the work become dull and uninteresting. He then goes on to some new ideas and after a time comes back to the study of the first idea with fresh pleasure in recognizing an old friend in new surroundings and with an increased appreciation of its full content. The old style course, by its intensive method, makes as great (if not greater) demands upon a student's power of assimilation, for it requires him to learn the new idea at once, while the unified course permits him to become acquainted with a new idea gradually, by meeting it at several different times, as illustrated by the example of the tangent line; and each time something is added to his previous knowledge, which is thus given time for a natural growth.

One other criticism of the unified course occurs to the writer at this time — the difficulty of a student referring to a text arranged for a unified course, when at some later time he wishes to use some formula or rule which he has forgotten. This difficulty, however, can easily be met by the compiling of a good index, and every student ought to learn the use of such an index, instead of shuffling the pages of his book till he finds what he wants. If the student of a unified course wishes to refer to some of the special texts of algebra, geometry, calculus, etc., it might be possible that he would not know what texts to consult — not know whether he ought to look in an analytic geometry or a calculus. But this difficulty seems so unlikely that it is hardly worth considering.

It is not to be assumed that all mathematics through the first course in calculus should be arranged in a single unified course. In fact, the break in going from the secondary school to the college seems to be an argument against such a large consolidation and to point to the desirability of at least two such unified courses.

Finally, such unified courses in the mathematics through the first course in Calculus are directly in line with what has long been done in the higher mathematics, as shown by the courses in analysis given by the French, and to some extent by the Germans, for advanced students. Surely a method which is so valuable there ought to be considered for other grades of work, unless the teaching of elementary mathematics is regarded as perfect at the present time; and the lively discussion of questions of teaching by the various associations of teachers makes one believe that the teachers are not so content with the present results.

JORDANUS NEMORARIUS AND JOHN OF HALIFAX.

By L. C. KARPINSKI, Teachers College, Columbia University.

Cantor closes the first volume of his *Geschichte der Mathematik*[1] with the names of Leonard of Pisa and Jordanus Nemorarius, stating that these men mark a new period in the development of mathematical science. Cantor's discussion[2] of the part played by Jordanus in opening a new era rests largely on the Schonerus edition[3] of an anonymous mediaeval *Algorismus Demonstratus* which was long attributed to Regiomontanus, for no other reason apparently than that Regiomontanus made a copy of a Vienna manu-

1). Vol. I, 3d edition, p. 911.

2). Vol. II, 2d edition.

3). D. E. Smith, *Rara Arithmetica*, pp. 178-179. Curtze (note, p. 20, Einige Materialen zur Geschichte der Mathematischen Facultaet der alten Universitaet Bologna, S. Gherardi, translation by Max. Curtze, Berlin, (1871), states that there is a XIV cent. Ms. in the library of Basle.

script of the same, which copy was used by Schoener for his edition[1]. The text is now attributed to an unknown Magister Gernardus[2]. Enestroem states[3] that the *Algorismus demonstratus* seems to have been unknown to the older writers on the history of mathematics. In a later article[4] he points out that the question as to which Algorismus Jordanus wrote seems to be very complicated, since there are several extant which are attributed to him[5].

In looking over the work of Wallis[6] on the early history of our numerals, chapter IV, *Quam antiquus sit harum in his regionibus Figurarum Numeralium usus*, I chanced upon a reference to an *Algorismus demonstratus* by Jordanus which is of sufficient importance to warrant giving the extract in full[7]. It should be noted that Wallis was a scholarly writer upon whose information we are able to rely. This reference antedates the work of Chasles,[8] mentioned by Enestroem as the first notice, by over one hundred and fifty years. The quotation is taken from the English edition of the Algebra which appeared in 1685.

"We have also, in Manuscript," says Wallis, "another treatise of *Algorism*, of *Jordanus*, (whom Vossius placeth about the year 1200 and contemporary with that *Campanus*, who wrote *De Computo Ecclesiastico*;) entitled *Algorismus Jordani, tam in Integris quam in Fractionibus, demonstratus*;[9] in which, the use of these Figures, and the way of numbering by them, is with great accuracy described and demonstrated. Which *Algorism* of his is very different from his *Arithmetica*, published and illustrated by *Faber Stapulensis*; yet so, as it may very well be judged, by his manner of demonstration, to be a work of the same man. And the Manuscript itself, as appears by the hand and by the shape of the Figures, is very ancient. And in the same Manuscript book, wherein that of *Jordanus*, and some other small pieces are written, I found at the end of it two Celestial Schemes, relating to the year 1216; the one of them is called *Figura Anni*, representing the position of the Heavens on *March* 22, 1216; the other, *Figura Conjunctionis Saturni et Martis*, shewing the position of the Heavens at the time of that Conjunction which happened the same year, *October* 4, 1216. They are both of them described by these Numerical

1). G. Enestroem, *Ist Jordanus Nemorarius Verfasser der Schrift "Algorithmus Demonstratus?"* Bibl. Math. 5_3, pp. 9-14.

2). P. Duhem, Sur l'Algorithmus Demonstratus, Bibl. Math. 6_3, p. 13.

3). L. c., note p. 9.

4). *Uber die "Demonstratio Jordani de Algorismo."* Bibl. Math. 7_3, pp. 24-30.

5). See also G. Enestroem, *Uber zwei angebliche mathematische Schulen im christlichen Mittel alter*, Bibl. Math. 7, pp. 252-262, and *Uber eine dem Jordanus Nemorarius zugeschrievene kurze Algorismusschrift*, Bibl. Math. 8_3, pp. 135-153. P. Treutlein, Abhandlungen zur Geschichte d. Math. Vol. II.

6). Johannis Wallis, Opera Mathem., Vol. II, Oxford, 1693.

7). In a personal letter from Mr. Enestroem to the author he states that this book is still in existence, Codex Savil. 21, Oxford, and that it was examined by Professor Bjornbo in 1906. It contains the text of the *Demonstratio Jordani de Algorismo.*

8). Apercu historique etc., Brussels, 1837.

9). The Latin version, Opera Omnia, vol. II, p. 13, is *Algorismus Jordani, tam in integris quam in fractis, demonstratus.* It adds also *Codex Saviliensis.*

Figures; and, in likelihood, were calculated about that time, in order to some Astrological Predictions to be made thereupon. And it so happens, that this last page of that Piece, proves to be the latter leaf of that same piece of Parchment, which begins that Book of *Algorismus Demonstratus* and therefore later written than it."

This statement by Wallis further complicates the matter under discussion,[1] as in the other works ascribed to Jordanus there is no special mention in the title of fractions, and in the treatment of fractions the name is given either as minutiæ philosophicæ or phisicæ (Sexagesimal fractions) and minutiæ vulgares (common fractions).

At the same time it would seem to show that some Jordanus did write an Algorismus about the beginning of the thirteenth century. Of men by the name of Jordanus several of about this time are well known in Church History[2] and, doubtless, there were others outside of the church. Jordanus Nemorarius is identified by Cantor with Jordanus de Saxonia (c. 1190-1237) of the order of Dominicans, based partly upon some early chronicles and partly upon the fact that this work often appears in the same volume with other works which are quite certainly the works of the Dominican. The name Nemorarius nowhere appears in the letters or writings of Jordanus of Saxonia, and the derivation of the appellation Nemorarius from his birthplace[3] rests upon an error in assuming the same to be Borrenstrick.

If the Saxon Jordanus did write any of these algorisms it is highly probable that this work had considerable influence in spreading the knowledge of our numerals throughout Europe, for as head of the order of Dominicans this man made many journeys over all of Europe; indeed he perished in a shipwreck in February, 1237, on the return voyage from a trip to the Holy Lands.[4] He had a reputation as a brilliant orator and is said to have made a thousand converts to his order. It would indeed be rather surprising if, in his wanderings, he had not somewhere learned and discussed the new symbols which were then coming into use.

Concerning the influence and even the identity of Jordanus we are much in doubt. In regard to the influence of John of Halifax there is no doubt. Maximilian Curtze,[5] who made a special trip through European libraries in search of Mathematical MSS., found that there are forty-five copies of the Algorismum of Sacrobosco in the three libraries of Munich, Vienna, and Erfurt (Amploniana). The Coxe catalog of an Oxford library gives ten further and there are doubtless numerous others extant in Europe. Of those mentioned at least seven are of the thirteenth century and the

1). See, however, note 7 above.

2). Potthast, *Bibliotheca Historica Mediaevi*, mentions four who lived between 1175-1300.

3). By Cantor, vol. II, J. J. Bertheir, *B. Jordanis de Saxonia, Opera, Friburg*, 1891, states that J. was born in Westphalia, not Borrenstrick. L'abbe E. Bernard, *Les Dominicains dans l'Universite de Paris*, Paris, 1883, largely on Jordanus.

4). Acta Santorum, Feb. 2, vol. II, pp. 720 ff.

5). *Petri Philomeni de Dacia in Algorismum vulgaren Johannis de Sacrobasco Commentarius*, Copenhagen, 1897, Max. Curtze.

others of later date up to the sixteenth century. In America there are two copies, one in the Plimpton Collection[1] and the other in the Columbia University Library. Both of these copies are the work of the fifteenth century.

In addition to the Algorismus of John of Halifax there are numerous commentaries on this work extant of which some also go back to the thirteenth century. Of these the one which seems to have been most widely used was that made by Petrus de Dacia in 1291. Of this commentary[2] five copies are mentioned by Enestroem[3] and five further by Curtze in the work cited. One Petrus de Dacia is given[4] as Rector of the University of Paris in 1326 and he may very well have been the author of this work. Petrus de Dacia wrote further a multiplication table from 1x1 up to 49x49,[5] a *Tabula de Loco Lunae,*[6] a Calendar and also a work "de calculo seu computo" which was probably on the computation of Easter.[7]

John of Halifax (Holywood, Holywod or Holiwod, Halifax, Holifax, Holybush, Holywolde Sacro-Busto or -Bosco, Sacrobosco, Sacrobuschus or Sacrobusto[8]) was born at Halifax, Yorkshire, studied at Oxford and later wandered to Paris where he lectured on mathenatics and astronomy. His most famous work was probably the *Sphaera mundi,* also known as *Libellus de Sphaera,* a treatise on astronomy which was used as a text-book in the Universities of the Middle Ages for several centuries.[9] He wrote further a *Computus Ecclesiasticus* and this *Algorismus,* which appears now to have been quite as popular from 1250-1550 as was the work on astronomy. The Algorismus was first published in Strassburg in 1488[10] and numerous editions followed. The text has been published also under the title *Tractatus de Arto Numerandi*[11] and *Oposculum de praxi numerorum quod Algorismum vocant.*[12] Copies of this work have been frequently attributed to other writers. The Plimpton Library MS. was ascribed to Mugling until Professor Smith made an examination of the text, while more recently an *Algorismus Doctrinalis* in the Hunterian Museum at Glasgow[13] is attributed to Michæl, Monk of Dover. As the opening and closing lines which are given are identical with lines in Sacrobosco's *Algorismus* it is seen to be the work of the same man, and this text is probably a student's copy made by this Michæl.

Nor is it to be suspected at all, that any attempt was made by the

1). D. E. Smith, *Rara Arithmetica.*

2). Text by Max Curtze, l. c.

3). In publications of the Royal Danish Academy.

4). Potthast, Bibl. Historica Mediiaevi, who also mentions a Petrus de Dacia, Monachus Vistyensis, lector Skeningae, a. 1271-1275.

5). Enestrom, Bibl. Math., 1890, p. 32.

6). *A Catalog of the Manuscripts in the Library of the Hunterian Museum* in the University of Glasgow, begun by John Young and completed by P. H. Aitken, Glasgow, 1908. Other copies are in the British Museum, additional MSS., No. 35, 317, f. 1, in Oxford Univ. Lib. (Catalog of MSS., Ashmole, 1845), and in Paris.

7). Vossius, De Universae Mathesios Natura et Constitutione, Amstel., 1650.

8). Vossius, l. c., Smith, *Rara Arithmetica.*

9). Aschbach, J., *Gesch. d. Wiener Universitaet im erster Jahrhunderte ihres Bestehens,* Wien, 1865, p. 93.

10). Smith, l. c., pp. 31-33.

11). Halliwell, *Rara Mathematica.*

12). Paris, 1510, Clichtoveus edition.

13). Young and Aitken, l. c., pp. 388-389.

copyists of this work to take the credit for its composition. Many of the copies found are, doubtless, simply students' lecture notes which were, of necessity, as the use of this work was largely before the invention of printing, written out. Even a slight examination of the Columbia MS. and the Plimpton MS. shows such marked divergence in phraseology from the text published by Curtze (and from each other) that we are led to conclude that these were lecture notes largely for the personal use of the writer. The shorthand character of the writing, making their reading practically impossible in many places, further confirms this view.

This *Algorism* of Sacrobosco is of interest for the reason, also, that it is probably due to the extended use of this work that the appellation Arabic numerals became common. In two places there is mention of the inventors of this system. In the introduction it is stated that this science of reckoning was given out by a philosopher named Algus whence the name *Algorismus*. Petrus de Dacia recognizes the Arabic origin of the word Algos,[1] stating that this is the name of some Arabic philosopher. The Columbia manuscript gives also the derivation of *Algorismus* from *ars+rismos* which word means number. In the section on numeration reference is made to the Arabs as the inventors of this science.[2] While some of the commentators knew of the Hindu origin, e. g., Petrus de Dacia,[3] most of them undoubtedly took the text as it stood and so the Arabs were credited with the invention of the system.

In conclusion it may be well to state that this and the *Algorismus* of John of Lima are the two of the oldest works known which bear the name of al-Khowarazmi, for the work of Frater Sigsboto which was published by Curtze under the title *Uéber eine Algorismus-schrift des XII Jahrhunderts*[4] nowhere uses the term *Algorismus*. There is also the Saliminian Codex[5] of the end of the twelfth century (textual evidence of date, as it bears no date nor name of author), and the *Carmen de Algorismo*[6] of Alexander de Villadei of which some lines are quoted by Sacrobosco.

John Bale, who made an examination of English Libraries before 1548,[7] gives Marianus Scotus (1208-c. 1382) as the author of an Algorism. In the absence of any further knowledge of the MS. lost since, at least, 1748,[8] we cannot place much emphasis upon this statement. At the same time it is well to note that another *algorismus* of about this same time is

---

1). "Quidam philosophus editit nomine Algus, unde et *Algorismus* nuncupatur," Curtze edition, l. c. p. 1. "Algorismus ab algos," Columbia MS. "Edidit Algus," also "Explicit Algorismus" at end, Plimpton MS.

2). "Sinistrorsum autem scribimus in hac arte more arabico sive indaico, huius scientiae inventorum," Curtze edition, p. 3. The Plimpton MS. omits the words "sive indaico."

3). "Non enim omnis numerus per quas cum que figuras in dorum repraesentatur," Curtze edition, p. 25.

4). *Abhandlungen zur Geschichte der Mathematik*, vol. VIII.

5). Index Britanniae Scryptorum, Joannes Balens, edited by R. L. Poole, Oxford, 1902, p. 287, "Ee collegio Martonensi, Oxon.

6). Halliwell l. c. This Carmen was widely used as shown by the large number of manuscript copies found in European libraries.

7). Index Brit., Script., 1548.

8). Tanner, Bibl. Britanno Hibernica, London, 1748.

credited to Guido Arezzo by Wailly[1] on the authority of the Benedictines, and also in the Nouveau traitè Diplomatique, in which the work is given as a computation on the sand table. This is the more likely inasmuch as several Arabic works written before 980[2] refer to calculation on a table, and to calculation without erasure. There is also a Computus of the XI century by Marianus in the Trinity College Library, bound with some arithmetic of the XII century.[3]

REVIEW OF A SIGNIFICANT TEXT IN CALCULUS.*

By N. J. LENNES, Massachusetts Institute of Technology.

This little book, entitled Elements of Differential and Integral Calculus, deserves more than a passing notice, especially because it is intended for a class of students differing considerably from those who now study elementary calculus in our colleges and universities. As stated in the preface, it is the result of a series of lectures delivered during the last six years to classes consisting chiefly of students of engineering and chemistry. In working over the lectures "it soon appeared," continues the preface, "that the mathematical knowledge which needs to be possesseed by a student before attempting the calculus is very much less than has been supposed. For example, the binomial theorem of algebra and the addition theorem of trigonometry are quite unnecessary. This book is written with the view of making the subject more easily and generally accessible than it has been hitherto. The principles of the differential and integral calculus ought to be counted as a part of the intellectual heritage of any educated man or woman in the twentieth century no less than the Copernican system or the Darwinian theory. In order to make a beginning no previous knowledge of mathematics is needed beyond the most elementary notions of geometry, a little algebra, including the law of indices and the definitions of the trigonometric funetions."

The general mode of presentation bears striking evidence of the influence of recent discussions of mathematical pedagogy. The consideration of each new process is preceded by concrete and simple problems in the solution of which the process is required. The simplicity of the problems themselves leaves the mind free to concentrate upon the one new thing offered for consideration, namely, the process of solution.

1). Elements de Paleographie, Paris, 1838, vol. I, pp. 711–716.

2). *M. Suter, Abhandlungen zur Geschichte der Mathematik, Fihrist*, pp. 37, 40, 41.

3). *Trinity College Library, Catalog of Manuscripts.*

*Elements of the Differential and Integral Calculus by A. E. H. Love. Cambridge University Press, pages 204+XIII.

The subject of coordinate geometry is introduced by such problems as the representation of the relation between the length of a spring and the weight of a body suspended from it, the relation between the readings of Centigrade and Fahrenheit thermometers, and the distances covered by a falling body in different intervals of time. The notion of "gradient" or "slope" is emphasized first in connection with the straight line and then in connection with the parabola obtained from the falling body problem. Numerous little exercises are interspersed for finding the gradient of straight lines and simple curves.

Chapter II deals with differentiation more formally under the following heads:

(a) Falling bodies.
(b) Speed of advancing body.
(c) Rates of change in general.
(d) Tangents to a curve.

Up to this point not a word has been said as to what is meant by "limit," though the word and the idea have been freely used. We now read (p. 21): "We shall be able to proceed more quickly afterward, and we shall be more certain that our work is correct, if we take a little time to think exactly what it is that we mean when we say that a function of h tends to a limit as h tends to zero." Then follows a beautifully simple exposition. Thus at every point the student is taken fully into the confidence of the author.

After the proofs of the formula for differentiating u^n for all rational values of n the second derivative is introduced and this is followed immediately by applications to tangents, approximations such as the use of $x^n+nx^{n-1}h$ for $(x+h)^n$, and more generally the use of $f(x)+hf'(x)$ for $f(x+h)$, maxima and minima including the use of $f'(x)$ to discriminate between a maximum and a minimum, and the mean value theorems.

Chapter IV (p. 55) first introduces integration. Practically throughout the book integration is regarded as the inverse process of differentiation and not as finding the limit of a sum. To find an area it is first asked "what is the derivative of the area?" and similarly for finding volumes, surfaces, lengths of areas.

Chapter VI, which deals with logarithms and exponential functions, is more complicated and difficult than the rest of the book, as are also the problems and applications of this chapter.

Chapter VII consists of a treatment of trigonometric functions. The proofs are unique inasmuch as little or no use is made of trigonometric formulas beyond the mere definitions of the functions. The theorems of this chapter are immediately applied to problems in oscillatory motion.

Chapter VIII deals with methods of integration and Chapter IX with applications to length of curves, curvature, and areas of surfaces of revolution.

In Chapter X (p. 157) the definite integral is considered as a limit of a sum, but not even here is it defined as such a limit. It is simply shown that the definite integral as previously defined is indeed the limit of a certain sum.

Chapter XI deals with centers of gravity, centers of pressure and moments of inertia. Each of these is regarded primarily as a limit of a sum.

The Appendix (pp. 179-204) contains formal proofs of certain propositions which in the text are made evident by rather informal discussion.

What is needed of trigonometry and analytic geometry is introduced as required. It is rather surprising to find that the 178 pages of the body of the text could surely not have been shortened by more than 25 pages if full knowledge of these subjects had been assumed.

For years past Klein in Germany, Perry in England, Moore in the United States, and hosts of others have been urging upon the mathematical fraternity the possibility and desirability of introducing the notions of the calculus at an earlier stage than is now done. In view of this, the little book under review is particularly significant. It gives a concrete means of judging whether such simplifications as would be demanded by distinctly less mature classes would be possible. The reviewer dare not undertake to say that this is an ideal book for very young classes, or even as well fitted to the American situation as we might reasonably expect were the book written expressly for us; but barring a few minor features closely related to English usage where it differs from ours, and other features which are related to the particular class of students for which it was primarily developed, it seems that the grade of difficulty is adapted to the fourth year of our secondary schools or the first year in college.

Would not a one year course of college mathematics be vastly more interesting and instructive if based upon a book of this type than the present combination of trigonometry, college algebra, and analytic geometry? In a majority of our secondary schools we now have only three or at most three and a half years of mathematics. Would not such a course as this furnish admirable material with which to complete a full four years course?

Unfortunately the typography of the book is ineffective and the pages are monotonous in appearance. The page architecture is particularly bad. An important figure frequently occurs near the bottom of a right hand page with nearly all the discussion on the next page. But such minor defects aside, one may reasonably hope that this little volume points the way to a vitalizing of an important part of our mathematical curriculum.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

335. Proposed by PROFESSOR L. E. DICKSON, Ph. D., The University of Chicago.

A person has \$1800 in notes payable \$18 monthly, bearing 10% interest. Find their present value if the interest is payable at the maturity of each note; also present value if interest is payable annually. [An actual business transaction.]

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

From the reading of the problem, it is supposed that the \$1800 is the actual amount to be paid, including interest.

I. Let x=present value.

$$\text{Then } x=\frac{18}{1.00\frac{5}{6}}+\frac{18}{(1.00\frac{5}{6})^2}+\frac{18}{(1.00\frac{5}{6})^3}+\ldots+\frac{18}{(1.00\frac{5}{6})^{100}}$$

$$=\frac{18}{(1.00\frac{5}{6})^{100}}[1+1.00\tfrac{5}{6}+(1.00\tfrac{5}{6})^2+\ldots(1.00\tfrac{5}{6})^{99}]$$

$$=\frac{18}{(1.00\frac{5}{6})^{100}}\left(\frac{(1.00\frac{5}{6})^{100}-1}{.00\frac{5}{6}}\right)=2160\left(1-\frac{1}{(1.00\frac{5}{6})^{100}}\right)$$

$=2160(1-.436114)=1217.994$, the number of dollars.

II. In such transactions the interest for each year is calculated as though no payments were made. Hence,

$$x(1.10)^8(1.03\tfrac{1}{3})-216[(1.10)^7+(1.10)^6+\ldots+1](1.03\tfrac{1}{3})-72=0.$$

$$x(1.10)^8(1.03\tfrac{1}{3})=2160[(1.10)^8-1](1.03\tfrac{1}{3})+72.$$

$$x=2160\left(1-\frac{1}{(1.10)^8}\right)+\frac{72}{(1.10)^8(1.03\frac{1}{3})}.$$

$$x=\$1184.846.$$

336. Proposed by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

Evaluate the determinant

$$\Delta=\begin{vmatrix} a_1^2, & a_2^2, & a_3^2, & \ldots & a_n^2 \\ a_2^2, & a_3^2, & a_4^2, & \ldots & a_{n+1}^2 \\ a_3^2, & a_4^2, & a_5^2, & \ldots & a_{n+2}^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n^2, & a_{n+1}^2, & a_{n+2}^2, & \ldots & a_{2n-1}^2 \end{vmatrix}$$

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Subtract each row from the one which follows it, beginning with the last but one. Repeat the same operation, stopping at the second row. Keep repeating this operation, leaving out a row each time, until all the rows have been thus omitted; then if D=value of determinant and

$$\triangle^r a_s^2 = \triangle^{r-1} a^2_{s+1} - \triangle^{r-1} a_s^2,$$

we get

$$D = \begin{vmatrix} a_1^2, & a_2^2, & a_3^2, & \dots & a_n^2 \\ \triangle a_1^2, & \triangle a_2^2, & \triangle a_3^2, & \dots & \triangle a_n^2 \\ \triangle^2 a_1^2, & \triangle^2 a_2^2, & \triangle^2 a_3^2, & \dots & \triangle^2 a_n^2 \\ \dots & \dots & \dots & \dots & \dots \\ \triangle^{n-1} a_1^2, & \triangle^{n-1} a_2^2, & \triangle^{n-1} a_3^2, & \dots & \triangle^{n-1} a_n^2 \end{vmatrix}$$

Repeating the same series of operations on the columns, we get

$$D = \begin{vmatrix} a_1^2, & \triangle a_1^2, & \triangle^2 a_1^2, & \dots & \triangle^{n-1} a_1^2 \\ \triangle a_1^2, & \triangle^2 a_1^2, & \triangle^3 a_1^2, & \dots & \triangle^n a_1^2 \\ \triangle^2 a_1^2, & \triangle^3 a_1^2, & \triangle^4 a_1^2, & \dots & \triangle^{n+1} a_1^2 \\ \dots & \dots & \dots & \dots & \dots \\ \triangle^{n-1} a_1^2, & \triangle^n a_1^2, & \triangle^{n+1} a_1^2, & \dots & \triangle^{2n-2} a_1^2 \end{vmatrix}$$

If a_r^2 is a function of r of the pth degree in r, whose highest term has a coefficient unity, the quantities a_1^2, a_2^2, a_3^2, ... form an arithmetic series of the pth order.

If $p=n-1$ all the elements below the second diagonal vanish, while all those in it are equal to $(n-1)!$, and $D=(-1)^{n(n-1)/2}\,[(n-1)!]^n$.

If $m<(n-1)$, $D=0$.

These determinants have been called orthosymmetrical.

GEOMETRY.

361. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

$ABCD$ is a quadrilateral. The bisectors of A and C meet in O_1; those of B and D meet in O_2. Find the tangent of the angle between AD and O_1O_2 in terms of sines and cosines of A, D, $A+B$, and $A+D$.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let $ABCD$ be the quadrilateral. Produce AB, DC, and AD, BC until they intersect in E, F, respectively. Take ADE as the triangle of reference for trilinear coordinates.

Let $\gamma=0$, be the epuation to AD; $\beta=0$, the equation to AB; $\alpha=0$, the equation to DC; $l\alpha+m\beta+n\gamma=0$, the equation to BC.

Also let $P=\sqrt{[l^2+m^2+n^2-2mn\cos A-2nl\cos D-2ml\cos E]}$.

(1) $\alpha-\gamma=0$, bisects angle D.
(2) $l\alpha+m\beta+n\gamma-P\beta=0$, bisects angle B.
(3) $\beta-\gamma=0$, bisects angle A.
(4) $l\alpha+m\beta+n\gamma-P\alpha=0$, bisects angle C.
(1) and (2) intersect in

$$\frac{\alpha_1}{P-m}=\frac{\beta_1}{l+n}=\frac{\gamma_1}{P-m}=\frac{2\triangle}{(a+c)(P-m)+b(n+l)}=O_2.$$

(3) and (4) intersect in

$$\frac{\alpha_2}{n+m}=\frac{\beta_2}{P-l}=\frac{\gamma_2}{P-l}=\frac{2\triangle}{a(n+m)+(b+c)(P-l)}=O_1.$$

Equation to O_1O_2 is

$$(5)\quad \alpha(P-l)+\beta(P-m)-\gamma(P+n)=0.$$

The angle between (5) and $\gamma=0$ is

$$\tan\phi=\frac{\sin A-\sin D+(l\sin D-m\sin A)/P}{1+\cos A+\cos D+(n-l\cos D-m\cos A)/P}.$$

But angle $(180^\circ-F)=(A+B)=$angle BC makes with AD.
$\therefore \sin(A+B)=(l\sin D-m\sin A)/P;$
$\cos(A+B)=(n-l\cos D-m\cos A)/P.$

$$\therefore \tan\phi=\frac{\sin A-\sin D+\sin(A+B)}{1+\cos A+\cos D+\cos(A+B)}.$$

362. Proposed by V. M. SPUNAR, M. and E. E., 3536 Massachusetts Avenue, N. S., Pittsburg, Pa.

Show that the focus of an ellipse may be regarded as an indefinitely small circle having double contact with the ellipse, the directrix being the chord joining the points of contact.

Solution by PROFESSOR F. L. GRIFFIN, Williams College.

A circle with its center at $(x_0, 0)$ any point of the major axis inside the evolute $[x_0<ae^2]$, and having for its radius the length of the normal which meets the axis in that point, is tangent to the ellipse at two points, say $(x_1, \pm y_1)$. From the equation of the normal to $b^2x^2+a^2y^2=a^2b^2$ at (x_1, y_1) we find, since $a^2-b^2=a^2e^2$, $x_0=e^2x_1$; or $x_1=x_0/e^2$. Also the normal length is given by $N^2=(x_1-x_0)^2+y_1^2$, which reduces to $N^2=(1-e^2)\times(a^2-e^2x_1^2)=(1-e^2)(a^2e^2-x_0^2)/e^2$. Thus the circle has the equation

$$(x-x_0)^2+y^2=(1-e^2)(a^2e^2-x_0^2)/e^2. \quad (1)$$

Let x increase toward the limit ae; then while the intersections are imaginary for $x_0>ae^2$, the analytical conditions for tangency are still fulfilled. For $x_0=ae$ the right member of (1) vanishes and the circle becomes the focus, since $x=x_0=ae$, $y=0$ are the only real solutions of (1). The value of x_1 has then become ae/e^2 or a/e, the abscissa of all points in the directrix.

Also solved by V. M. Spunar, G. B. M. Zerr, J. Scheffer, and Levi S. Shively.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

168. Proposed by A. H. HOLMES, Brunswick Maine.

Find integral values for x, y, u, and v from the following:

$$uv-xy=25x+29y+29u+29v-112.$$
$$3v-5u+5y-x=102.$$
$$4y-3v=419.$$

Solution by V. M. SPUNAR, M. and E. E., Pittsburg, Pa.

From (3), we have

$$v=\frac{4y-419}{3}\ldots(4),$$

which substituted in (2) and reduced will yield

$$u=\frac{9y-(x+521)}{5}\ldots(5).$$

Substituting (4) and (5) in (1), combining like terms, and we get

$$36y^2-7653y-x(19y-131)+326061=0.$$

$$\therefore x-\frac{36y^2-7653y+326061}{19y-131}=0, \text{ or } 19x-36y+\frac{140691y-6195159}{19y-131}=0, \text{ or}$$

$$361x-684y+140691-\frac{99277500}{19y-131}=0\ldots(6).$$

Hence, $99277500/(19y-131)$ must be an integer, and therefore $19y-131$ must be a factor of $99277500=2^2.3.5^4.7.31.61$. Thus

$$19y=131\pm(\alpha,\beta\ldots),$$

where $\alpha, \beta, \gamma, \ldots$ are the six different prime factors 2, 3, 5, 7, 31, 61 (with repetition 2 and 5) combined in products of 1, 2, 3, ..., 10 letters at a time.

By actual calculation and checking we have found the following ten solutions:

$x=-3343, 12721, 320, 305, 7547, -2225, -710, -3215, -8495, -56950;$

$y=2, 8, 29, 374, 4187, -1, -91, -1486, -4276, -29851;$

$u=568, -2634, -116, 454, 5923, +337, -126, -2136, -6102, -42446;$

$v=-137, -89, -101, 319, 5473, -141, -261, -2121, -5841, -39941.$

Also solved by G. B. M. Zerr.

169. Proposed by R. D. CARMICHAEL, Princeton University.

Let $Q_n(x)=0$ be the equation whose roots are all the primitive nth roots of unity without repetition. In $Q_n(x)=0$ replace x by α/β, a fraction in its lowest terms, and clear of fractions. Let $Q_n(\alpha, \beta)$ represent the resulting first member. Set $n=mp$ where p is the largest prime factor of n. It is required to find all the integral values of α, β, m, p satisfying the following relations:

$$(1) \quad Q_m p(\alpha, \beta)=p,$$
$$(2) \quad \alpha^m-\beta^m \equiv 0 \pmod{p}.$$

One such solution is: $\alpha=2, \beta=1, m=2, p=3$. (See MONTHLY, Vol. XII, p. 89.)

[No solution of this problem has been received.]

170. Proposed by PATRICK WALSH, 1451 Annunciation Street, New Orleans, La.

The areas of rectangles A and B are respectively 15170 10/27 and 31230.3627. Find the sides and diagonal of each rectangle in exact or rational numbers.

Solution by B. F. FINKEL, Ph. D.

For A, let x and y be the dimensions of the field. Then

$$xy=15170\tfrac{10}{27}=\frac{2^{14}\cdot 5^2}{3^3} \ldots (1), \text{ and } \sqrt{(x^2+y^2)}=d \ldots (2),$$

where d is the diagonal, which is to be rational. Solving (1) for y and substituting the value thus found in (2) and reducing, we have

$$\frac{\sqrt{(3^6x^4+2^{28}\cdot 5^4)}}{3^3y}=d.$$

Let $x=\frac{2^7\cdot 5}{3^2}z$. Then $d=\frac{2^7\cdot 5}{3^2z}\sqrt{(z^4+9)}$. Let $\sqrt{(z^4+9)}=z^2t-3$. Then $z^2=\frac{6t}{t^2-1}$.

We must now find such values for t as will make $\frac{6t}{t^2-1}$ a perfect square. These values are 0, 1, 2, 3, $\frac{1}{2}$, $\frac{1}{3}$. For $t=2$, $z=2$,

$$d=\frac{2^6.5^2}{3^2},\ x=\frac{2^8.5}{3^2},\ \text{and}\ y=\frac{2^6.5}{3}.$$

For $t=3$, $z=\frac{3}{2}$, $d=\frac{2^6.5^3}{3^2}$, $x=\frac{2^6.5}{3}$, and $y=\frac{2^8.5^2}{3^2}$.

The values of 0 and 1 for t, give reciprocal limiting values for d, x, and y.

A similar treatment for B leads to the value, $z=2\sqrt{\frac{15t}{t^2-1}}$.

z is rational for $t=0$, 1, 4, $-\frac{1}{4}$, giving the values $z=0$, ∞, 4.
The values of x and y corresponding to $z=4$, are

$$x=\frac{3^2.19.179}{2^3\times5},\ y=\frac{3\times19.179}{5^2},\ \text{diagonal}=\frac{3^2.17.19.179}{2^3\times5}.$$

171. Proposed by PROFESSOR E. B. ESCOTT, Ann Arbor, Mich.

Solve completely:

$$\begin{aligned}2x^2-1&=y,\\ 2y^2-1&=z,\\ 2z^2-1&=w,\\ 2w^2-1&=x.\end{aligned}$$

I. Solution by PROFESSOR L. E. DICKSON, Ph. D., The University of Chicago.

It will be shown that the only integral solution is $x=y=z=w=1$.

If $x=0$ or ±1, then $y^2=1$, $z=1$, $w=1$, so that the fourth equation gives $x=1$.

Next, let $x^2>1$. Then $y>x^2$, $z>y^2>1$, $w>z^2>1$, $x>w^2$.

Hence, $x>x^{16}$, which contradicts $x^2>1$.

Also similarly solved by G. B. M. Zerr, and S. G. Barton.

II. Solution by the PROPOSER.

Let $x=\cos\phi$; then $y=\cos2\phi$, $z=\cos4\phi$, $w=\cos8\phi$, $x=\cos16\phi$.

Since $\cos16\phi=\cos\phi$, we have, $\cos16\phi-\cos\phi=0$, which may be written

$$-2\sin\tfrac{17}{2}\phi.\sin\tfrac{15}{2}\phi=0.$$

If $\sin\frac{17}{2}\phi=0$, $\phi=\frac{2n\pi}{17}$. If $\sin\frac{15}{2}\phi=0$, $\phi=\frac{2n\pi}{15}$.

Taking $\phi=\frac{2n\pi}{15}$, we get seven roots, viz,

$$x_1=\cos\frac{2\pi}{15}=\cos 24^\circ=\tfrac{1}{8}[\sqrt{(30-6\sqrt{5})}+\sqrt{5}+1];$$

$$x_2=\cos\frac{4\pi}{15}=\cos 48^\circ=\tfrac{1}{8}[\sqrt{(30+6\sqrt{5})}-\sqrt{5}+1];$$

$$x_3=\cos\frac{6\pi}{15}=\cos 72^\circ=\tfrac{1}{4}(\sqrt{5}-1);$$

$$x_4=\cos\frac{8\pi}{15}=\cos 96^\circ=-\tfrac{1}{8}[\sqrt{(30-6\sqrt{5})}-\sqrt{5}-1);$$

$$x_5=\cos\frac{10\pi}{15}=\cos 120^\circ=-\tfrac{1}{2};$$

$$x_6=\cos\frac{12\pi}{15}=\cos 144^\circ=-\tfrac{1}{4}(\sqrt{5}+1);$$

$$x_7=\cos\frac{14\pi}{15}=\cos 168^\circ=-\tfrac{1}{8}[\sqrt{(30+6\sqrt{5})}+\sqrt{5}-1).$$

Taking $\phi=\frac{2n\pi}{17}$, we get eight roots. Gauss has shown how the complex roots of $x^{17}-1=0$ may be found by solving a chain of quadratic equations. If ω is a complex root of $x^{17}-1=0$, then

$$\omega=\cos\frac{2\pi}{17}+i\sin\frac{2\pi}{17}, \text{ and } \omega^{16}=\cos\frac{2\pi}{17}-i\sin\frac{2\pi}{17},$$

from which we have

$$\cos\frac{2\pi}{17}=\tfrac{1}{2}(\omega+\omega^{16})=x_8;$$

and similarly,

$$\cos\frac{4\pi}{17}=\tfrac{1}{2}(\omega^2+\omega^{15})=x_9,$$

$$\cos\frac{6\pi}{17}=\tfrac{1}{2}(\omega^3+\omega^{14})=x_{10},$$

$$\cos\frac{8\pi}{17}=\tfrac{1}{2}(\omega^4+\omega^{13})=x_{11};$$

$$\cos\frac{10\pi}{17}=\tfrac{1}{2}(\omega^5+\omega^{12})=x_{12};$$

$$\cos\frac{12\pi}{17}=\tfrac{1}{2}(\omega^6+\omega^{11})=x_{13};$$

$$\cos\frac{14\pi}{17}=\tfrac{1}{2}(\omega^7+\omega^{10})=x_{14};$$

$$\cos\frac{16\pi}{17}=\tfrac{1}{2}(\omega^8+\omega^9)=x_{15};$$

We have to solve the chain of equations:

$$k^2+k-4=0. \quad \text{Roots } k_1,\ k_2.$$
$$l^2-k_1 l-1=0. \quad \text{Roots } l_1,\ l_2.$$
$$l^2-k_2 l-1=0. \quad \text{Roots } l_3,\ l_4.$$
$$4x^2-2l_1 x+l_3=0. \quad \text{Roots } x_8,\ x_{11}.$$
$$4x^2-2l_2 x+l_4=0. \quad \text{Roots } x_9,\ x_{15}.$$
$$4x^2-2l_3 x+l_2=0. \quad \text{Roots } x_{10},\ x_{12}.$$
$$4x^2-2l_4 x+l_1=0. \quad \text{Roots } x_{13},\ x_{14}.$$

These fifteen roots together with the root $x=1$ make the sixteen roots. From symmetry, the sets of values for y, z, and w, are the same as for x.

PROBLEMS FOR SOLUTION.

ALGEBRA.

337. **Proposed by I. M. CURTISS, Brooklyn, N. Y.**

Three regiments move north as follows: B is 20 miles east of A; C is 20 miles south of B, and each marches 20 miles between the hours of 5 a. m. and 3 p. m. A horseman with a message from C starts at 5 a. m. and rides north till he overtakes B, then sets a straight course for the point at which he calculates to overtake A, then sets a straight course for the next point at which he will again overtake B, then rides south to the point where he first overtook B, reaching that point at the same time as C, namely 3 p. m. What uniform rate of travel enabled the messenger to do this?

338. **Proposed by R. D. CARMICHAEL, Princeton University.**

Prove that $\pi=3+\frac{1}{3}\cdot\frac{1}{1.2}-\frac{1}{5}\cdot\frac{1}{2.3}+\frac{1}{7}\cdot\frac{1}{3.4}-\frac{1}{9}\cdot\frac{1}{4.5}+\ldots$

339. **Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Mich.**

Prove that if $a_1<2$ and $a_n=a_{n-1}^2-2$, $\frac{1}{a_1}+\frac{1}{a_1 a_2}+\frac{1}{a_1 a_2 a_3}+\ldots$
$=\frac{1}{2}[a_1-\sqrt{(a_1^2-4)}]$.

GEOMETRY.

368. Proposed by G. I. HOPKINS, Professor of Mathematics and Astronomy, Manchester High School, Manchester, N. H.

It is required to construct the triangle having given, base, vertical angle, and ratio of its altitude to sum of the other two sides.

369. Proposed by W. J. GREENSTREET, A. M., Editor, Mathematical Gazette, Stroud, England.

Prove by inversion that if two circles cut at a given angle, touch each a given circle, and pass each through the same fixed point, then shall the envelope of the points of contact be a conic.

370. Proposed by R. C. ARCHIBALD, Paris, France.

The trisectors of the angles of any triangle ABC are, in order, AF, AE, CE, CD, BO, BF. Show synthetically that D, E, F are the vertices of an equilateral triangle.

CALCULUS.

295. Proposed by C. E. FLANNAGAN, Wheeling, W. Va.

A hawk can fly v feet per second, a hare can run v' feet per second. The hawk, when a feet vertically above the hare, gives chase and catches the hare when the hare has run b feet. Find the length of the curve of pursuit. [Echols' *Differential and Integral Calculus*, page 253, Ex. 20.]

296. Proposed by C. N. SCHMALL, New York City.

Two currents C_1 and C_2 produce deflections ϕ_1, ϕ_2, respectively, in a tangent galvanometer. When is $(\phi_1-\phi_2)$ a maximum?

MECHANICS.

249. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

A load P is supported by three strings of equal size attached at the vertices of a triangle sides a, b, c lying in a horizontal plane. The load is vertically under the centroid of the triangle at a distance h from it. Find the stresses in the strings.

250. Proposed by C. N. SCHMALL, New York City.

A smooth circular table is surrounded by a smooth vertical rim. A ball of elasticity e is projected from a point at the rim in a line making an angle ϕ with the radius through that point. Show that the ball will return to the starting point after the second impact if

$$\tan\phi = \sqrt{\frac{e^3}{e^2+e+1}}.$$

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

174. Proposed by B. KRAMER, Student, University of Pittsburg, Pittsburg, Pa.

Find a general solution of $x(x+a)=y^2$, a, x, and y being integers. Given a, required to find x to satisfy conditions.

NOTES AND NEWS.

The former students of Professor Bolza at the University of Chicago, desiring to express in some tangible form their love and esteem, presented to him a beautiful loving cup at a dinner in his honor on June third. The following is quoted from the letter of presentation:

"Those among the students of Professor Bolza who know him best are foremost in their appreciation of his unusual qualities, both as a lecturer and in awakening the spirit of research. Faithful in precept and inspiring in example, he has been a tower of strength at the University for nearly two decades, while students by the hundreds, after drawing from him mathematical inspiration and power, have gone forth to all parts of this country, many of them to occupy positions of responsibility and trust in our leading colleges and universities. As a genial friend whose hospitality we have all enjoyed, as an inspiring teacher whose peer we have seldom known, as a contributor to mathematical science whose reputation is established here and abroad, as the man who with Professors Moore and Maschke made the University of Chicago from its earliest days one of the foremost of mathematical schools, we pay respectful tribute to Professor Bolza, and wish him and Mrs. Bolza the widest usefulness and the greatest happiness in their new "old home" in Freiburg."

At the University of Tennessee Professor John B. Hamilton has a leave of absence for the coming year and will spend the time in advanced study at the University of Chicago.

At Cornell University Dr. Virgil Snyder and Dr. J. I. Hutchinson have been advanced to full professorships.

At the University of Wisconsin Dr. Max Mason has been appointed to a full professorship in mathematical physics.

At the University of Kansas Dr. C. H. Ashton and Dr. Van der Vrees have been advanced to associate professorships, and Professor J. W. Young of the University of Illinois has been appointed head of the department of mathematics.

The next meeting of the National Education Association will be held in Boston during the week of July 28, 1910. The program of the Mathematical Section has been arranged as follows:

Why Do We Study Mathematics; A Historical and Philosophical Retrospect by Prin. Thomas J. McCormack, Chairman LaSalle-Peru Township High School, LaSalle, Illinois.

Discussion: (a) The Practical Limitations of an Ideal Course in American Secondary Mathematics, and the Educational Waste or Economy

in the Proposed Sequences of Studies, by Principal John Shaw French, Morris Heights School, Providence, R. I.

(b). Preliminary Report of the "National Geometry Syllabus Committee," and its Practical Pedagogical Implications, by William Betz, East High School, Rochester, N. Y.

(c). Applied Problems and the Role of Formal Drawing in Secondary Mathematics, by William Breckenridge, Stuyvesant High School, New York City.

The committee on the teaching of mathematics to students of engineering, which was appointed at the joint meeting of the Chicago Section of the American Mathematical Society and Sections A and D of the American Association for the Advancement of Science, held in Chicago, December 30-31, 1907, will present its report at the meeting of the Society for the Promotion of Engineering Education to be held at Madison, Wisconsin, June 23-25, 1910. There will be extended outlines presented for discussion on the following topics: (1), Geometry and Mensuration; (2), Algebra; (3), Trigonometry; (4), Analytic Geometry; (5), Calculus; (6), Theoretical Mechanics; (7), Numerical Computation and the Solution of Equations. All mathematicians are urged to be present at that meeting and to take part in the discussions.

At the University of California, Dr. D. N. Lehmer has been advanced to an associate professorship in mathematics.

Dr. Arnold Dresden, of the University of Wisconsin, will spend the coming summer in Europe.

Professor C. Runge, Kaiser Wilhelm exchange professor of mathematics at Columbia University for the present academic year, recently delivered five lectures on graphical methods, at the University of Michigan. His lectures on this subject, delivered at Columbia University, are to be published in book form by Columbia University. These methods have many important applications in astronomy, physics and engineering. M.

6—7

THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XVII. JUNE-JULY, 1910. NOS. 6-7.

MATHEMATICS BEYOND THE CALCULUS.

By G. A. MILLER, University of Illinois.

Students of mathematics generally find that no option as regards the order of subjects is open to them until they have completed an elementary course in calculus. The prescribed road generally is as follows: arithmetic, algebra, geometry, trigonometry, analytic geometry, calculus. After completing an elementary course in calculus the student who expects to become a mathematician frequently finds an entirely different situation since it often becomes necessary for him to decide not only as regards the order of courses but also as to what courses should be selected from the rich offering.

He may perhaps be aided by the fact that there are only three grand divisions of mathematics according to some of the best authorities, even if these divisions do not have any distinct boundaries. They are commonly named as follows: arithmetic and algebra, analysis, geometry. As every mathematician should know something of each one of these great fields, the student will naturally plan to take some comprehensive courses in each of them. To divide the time at his disposal into three approximately equal parts would be easy enough, but to divide it into three parts with due regard to his tastes and ability, and to the men who may be offering the respective courses often becomes a very complex problem.

It is evidently desirable that the student should not decide very early on the particular small field where he hopes to be better informed than any one else. He will probably find many suitable fields without looking for them. In due time he has to make his choice and after it is once made he should bear in mind that serious and honest work is even more important than the choice of the field, for mathematics as a whole will profit by a harmonious development of all the various fields, and those who can appreciate work in but one field are excelled only by the mathematical butterflies in their pernicious influence.

One of the first things that such a student should do is to acquire some fairly intelligent notion as regards the extent of the mathematical literature. The fact that there are about a hundred thousand articles and thirty-five